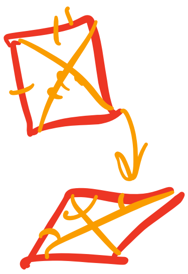
Similarities I



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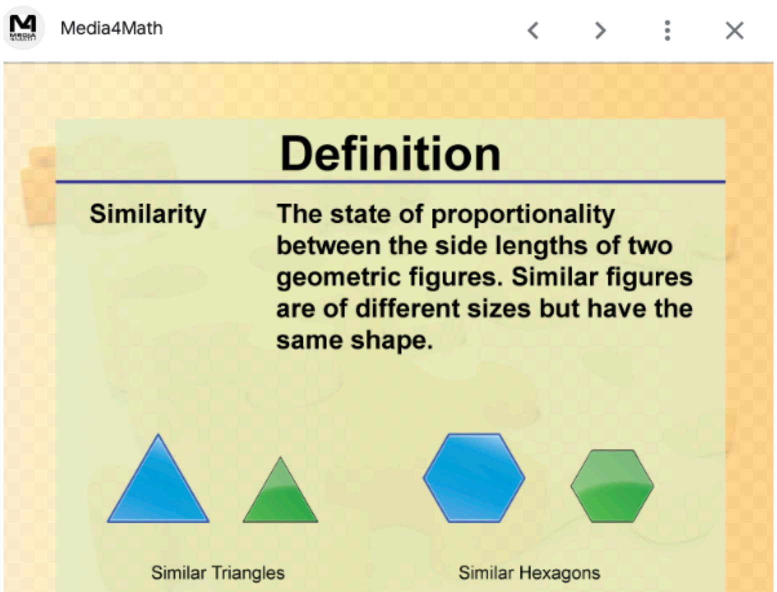


Similarity (geometry)

Article Talk Read

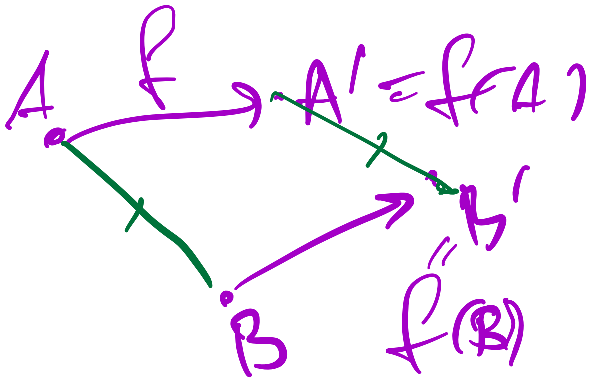
From Wikipedia, the free encyclopedia

and also a glide reflection

isometries preserve distances

In Euclidean geometry, two objects are **similar** if they have the same shape, or if one has the same shape as the mirror image of the other. More precisely, one can be obtained from the other by uniformly scaling (enlarging or reducing), possibly with additional translation, ~~rotation~~ and reflection. This means that either object can be rescaled, repositioned, and reflected, so as to coincide precisely with the other object. If two objects are similar, each is congruent to the result of a particular uniform scaling of the other.

Def 1 (1) Transformation f of the plane that preserves dist. btwn points i.e. $|f(A)f(B)| = |AB|$ is called a rigid motion or isometry



(2) Shapes obtained frm one another by an isometry are called congruent

Def 2 A similarity is a transformation of the plane that multiplies all distances btwn points by a fixed positive number, i.e. for any A, B

$$|f(A)f(B)| = k\,|AB|, \quad k>0$$

k is called the scaling or the coefficient of similarity

Observe, that if $k=1$, then f is an isometry, so every isom. is a similarity.

Now, is there anything else?

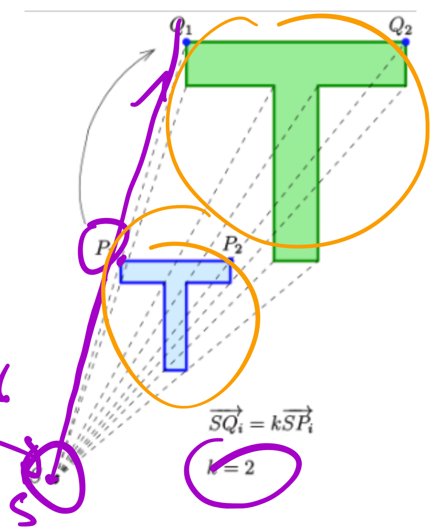
k can be < 0

$k \neq 0$

HOMOTHETIES

Def 3 A homothety with factor k and center C is the transf. sending pt A of the plane to A' on the line CA s.t. $|CA'| = |k| \cdot |CA|$ and on ray CA if $k > 0$ and on the opposite ray if $k < 0$

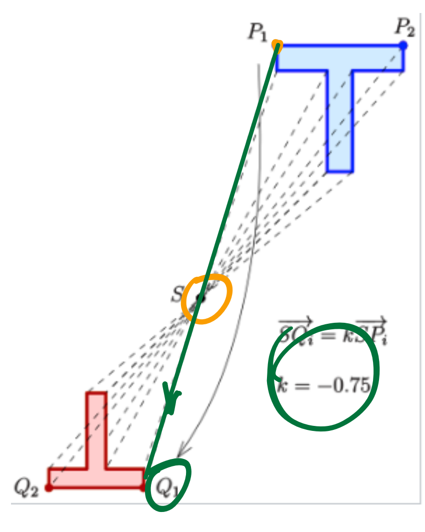
$H_S^2 \rightarrow$



$|SQ_1| = 2|SP_1|$

$Q_1 = f(P_1)$

c



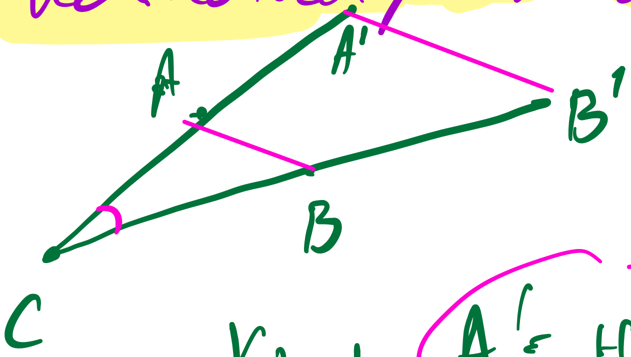
$\swarrow H_S^{-0.75}$

$k < 0$

Notation $f = H_S^k$

Thm A homothety H_C^k is a similarity with factor $|k|$

Pf



Know $A' = H_C^k(A)$, $B' = H_C^k(B)$

Need to show that $|A'B'| = |k| \cdot |AB|$

Indeed, we have $|A'C| = |k| \cdot |AC|$
$|B'C| = |k| \, |BC|$

So in ABC and $A'B'C$ we have proportional sides (with coeff $|k|$) and same angle btwn them

$\angle ACB = \angle A'CB'$

So $\triangle ABC$ & $\triangle A'B'C$ are similar with coeff $|k|$

So $|A'B'| = |k| \, |AB|$ $\boxed{\text{QED}}$

So H_C^k is indeed a similarity

Properties of similarities

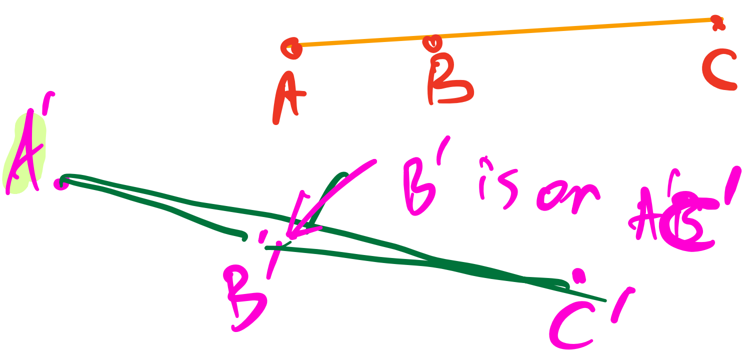
Let f be a similarity w factor k

Thm ①

1. f sends straight lines to straight lines
2. f sends circles to circles
3. f sends any polygon to a similar polygon (i.e. f preserves angles)

Proof

1. Let A, B, C be collinear



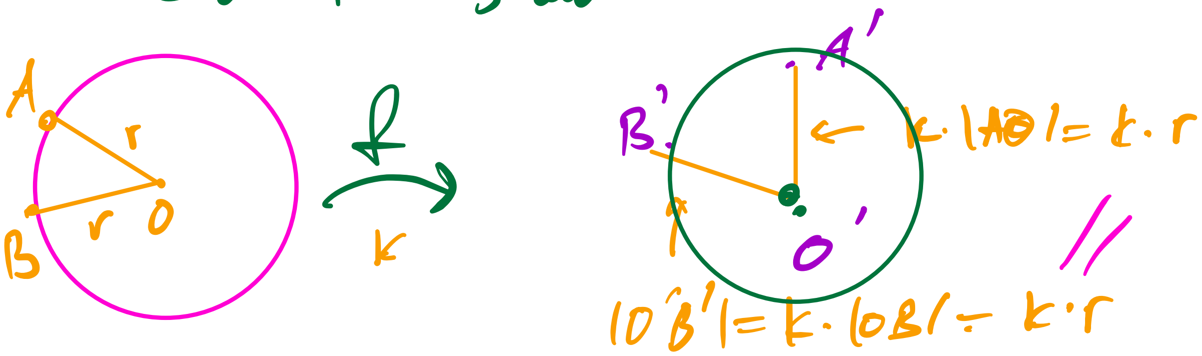
Need to show that $A' = f(A)$, B', C' are also collin (say B is betw A, C)

i.e. $|AB| + |BC| = |AC| \Rightarrow$

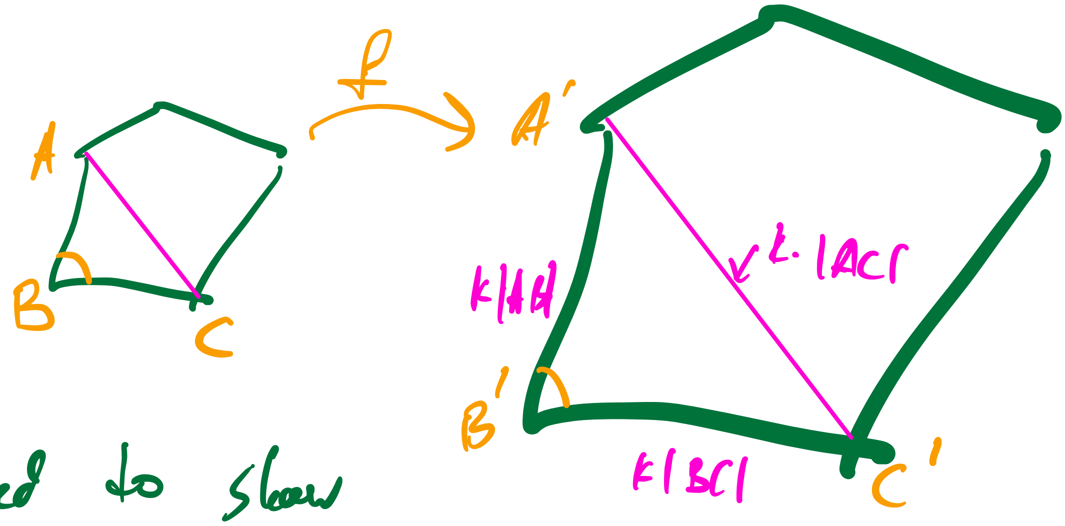
$$\underset{\text{is a sim}}{f} \Rightarrow \begin{cases} |A'B'| = k \cdot |AB| \\ |B'C'| = k|BC| \\ |A'C'| = k \cdot |AC| \end{cases} \Rightarrow |A'C'| = k(|AB| + |BC|) \Rightarrow |A'B'| + |B'C'|$$

2. Let A, B lie on a circle with center O and radius r

Need to show



3.



Need to show

$\angle ABC = \angle A'B'C'$

i.e. $\Delta A'B'C'$ is similar to ABC by SSS similarity.

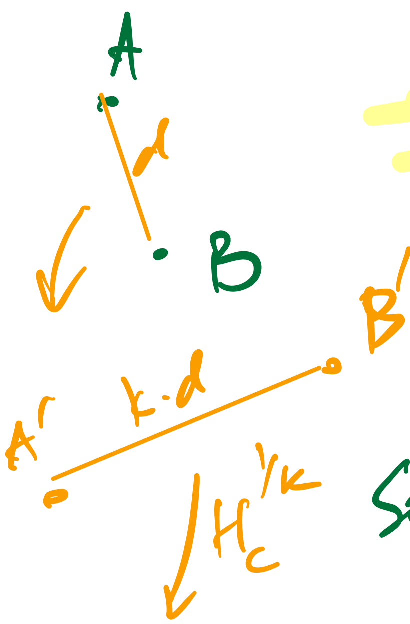
Thm 2 Every similarity f of factor k is a composition of a homothety H_C^k (with any C) and an isometry.

i.e. $f = H_C^k \circ g$, where g is an isom.

Pf Let f be this simil. of factor k, then take

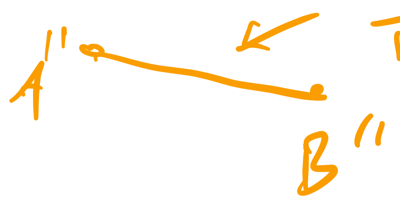
$$g = H_c^{1/k} \circ f .$$

scales by $\frac{1}{k}$ scales by k



So if $A' = f(A)$, $B' = f(B)$

$\frac{1}{k}|A'B'| = \frac{1}{k} \cdot k \cdot d = d = |AB|$



i.e. $|A''B''| = |AB|$

but $A'' = g(A)$, $B'' = g(B)$

so g is an isometry!

Prop 4 (important!)

If f_1 is a simil w factor k_1 and f_2 is a similarity w factor k_2, then

$f = f_2 \circ f_1$ is also a simil whose factor is $k = k_1 k_2$

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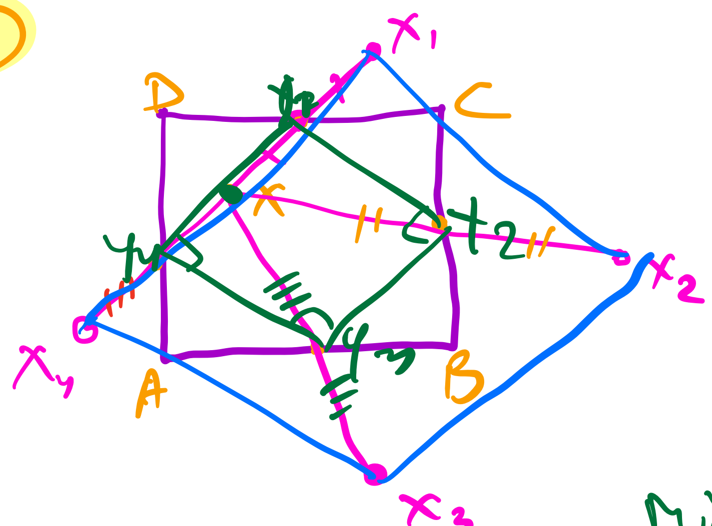
[Note This is (almost) saying that similarities form a group]

First applications

(1)



X - arbitrary pt in ABCD

Claim $X_1 X_2 X_3 X_4$ is a square

reflections of X about midpts of sides

Midpts $Y_1 Y_2 Y_3 Y_4$ form a square and the quadri' $X_1 X_2 X_3 X_4$ is obtained fm it by the homoth. H_X^2, so it is also a square bc H_X^2 is a similarity.

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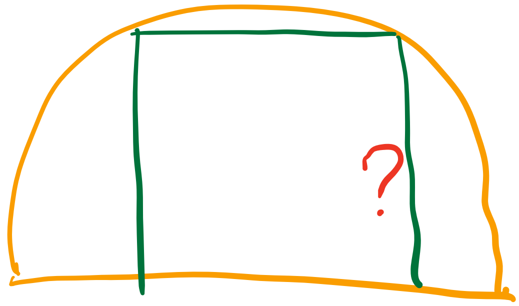
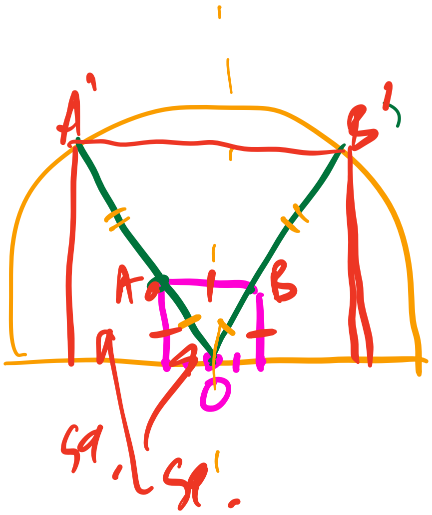
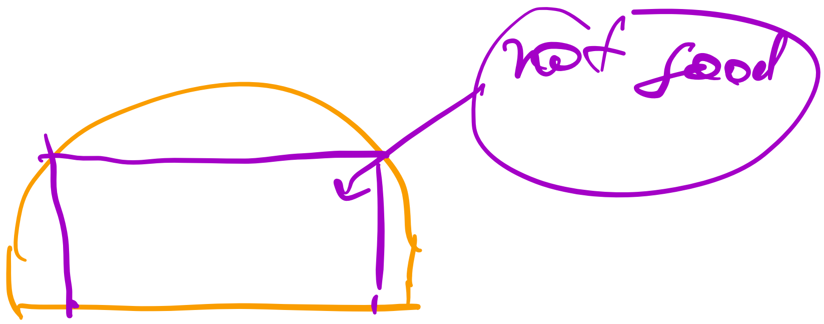
(2)

Inscribe a square in the given semicircle

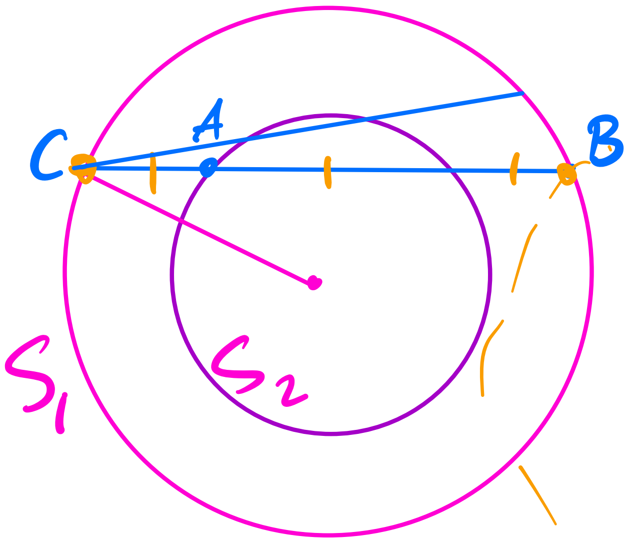






(3)

Given 2 concentric circles, draw a line on which they cut 3 segments of equal length



$\parallel H^3(S_1)$

S_3

$\frac{CB}{CA} = 3$, so $H_C^3(A) = B$

and $H_C^3(S_2) = S_3$

So $H_C^3(A)$ lands on S_3 $\rightarrow B$

i.e. B is an intersection of S_1 and of $S_3 = H^3(S_2)$